

Statistical properties of stochastic 2D Navier-Stokes equation

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Abstract

We investigate the conjecture suggested by numerical simulations and experimental evidence that the scaling exponents for the Navier-Stokes equation are the same as for a suitable linear advection equation. We prove the result for the Navier-Stokes equation with additive noise in a 2D spatial domain. To analyze the coupled system of the Navier-Stokes field u and the advection field w , one introduces a parameter λ which gives a symmetric system for (u^λ, w^λ) . This system is studied for any λ proving the well posedness and the uniqueness of its invariant measure μ^λ . Assuming universality of the scaling exponents to the force, the scaling exponents of u^λ and w^λ are the same. We prove that this is true also in the limit as $\lambda \rightarrow 0$, that is the 2D Navier-Stokes equation with additive noise has the same scaling exponents as the linear advection equation. Therefore, we have anomalous behaviour of the scaling exponents for the stochastic 2D Navier-Stokes equation.

1 Introduction

We consider the stochastic Navier Stokes equation

$$(1) \quad \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u = -\nabla p_1 + \frac{\partial W_1}{\partial t},$$

describing the motion of a fluid in an open bounded domain \mathcal{D} of \mathbb{R}^d . We assume that \mathcal{D} has a smooth boundary $\partial\mathcal{D}$ which satisfies the locally Lipschitz condition. Here, $u = u(t, \xi)$ is the velocity vector field, $p = p(t, \xi)$ is the scalar pressure field and $\nu > 0$ is the viscosity. $W_1 = W_1(t, \xi)$ is a Gaussian random field white in time, subject to the restrictions imposed below on the space correlation.

The velocity field u is subject to the incompressibility condition

$$(2) \quad \nabla \cdot u(t, \xi) = 0, \quad t \in [0, T], \xi \in \mathcal{D},$$

and to the boundary condition

$$(3) \quad u(t, \xi) = 0, \quad t \in [0, T], \xi \in \partial\mathcal{D};$$

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an initial condition is given

$$(4) \quad u(0, \xi) = u_0(\xi), \quad \xi \in \mathcal{D}.$$

Recently (see [1] and the references therein), it has been proposed that the Navier-Stokes equation and a relevant linear advection model

$$(5) \quad \frac{\partial w}{\partial t} - \nu \Delta w + (u \cdot \nabla)w = -\nabla p_2 + \frac{\partial W_2}{\partial t},$$

have the same scaling exponents of their structure functions, even if their statistics are different (i.e., in the "jargon" of stochastic PDE's their invariant measures are different). Here u is solution of equation (1); the noises W_1 and W_2 are independent and identically distributed.

We recall the definition of structure function (see [24]): it is an ensemble average of power of velocity differences across a length scale. The longitudinal structure function is

$$\left\langle \{[u(t, \xi + l\hat{\xi}) - u(t, \xi)] \cdot \hat{\xi}\}^p \right\rangle$$

with $\xi, \xi + l\hat{\xi} \in \mathcal{D}$, $p \in \mathbb{N}$ and $l \in \mathbb{R}$; $\hat{\xi}$ is a versor and we take the scalar product of vectors in \mathbb{R}^d . Similarly one can define the transverse structure function.

If the velocity field u is stationary in time, homogeneous and isotropic in space, the latter quantity depends only on the length scale l . We denote it by $S_u^p(l)$, that is

$$(6) \quad S_u^p(l) = \left\langle \{[u(t, l\hat{\xi}) - u(t, 0)] \cdot \hat{\xi}\}^p \right\rangle$$

for any t . For a turbulent field it is important to know how $S_u^p(l)$ depends on l for small l . The scaling exponents ζ_p are defined by

$$(7) \quad S_u^p(l) \propto l^{\zeta_p}$$

and are assumed to be universal in the limit $\nu \rightarrow 0$ when l lies in the inertial range $\eta_0 \nu^{3/4} \leq l \leq l_0$ (see e.g. [20, 24]); that is an equivalent definition is

$$(8) \quad \zeta_p = \lim_{l \rightarrow 0} \frac{\log S_u^p(l)}{\log l}.$$

Notice that $l \rightarrow 0$ implies also $\nu \rightarrow 0$.

Dimensional considerations provide $\zeta_p = \frac{p}{3}$. However, numerical and experimental results give $\zeta_3 = 1$, $\zeta_2 > \frac{2}{3}$ but close to the value $\frac{2}{3}$, and an anomalous behaviour $\zeta_p < \frac{p}{3}$ for $p > 3$. Therefore, the determination of the scaling exponents ζ_p is a crucial problem in turbulence theory (see [24]).

However, [1] provides numerical evidence that both the 2D Navier-Stokes equation and the Sabra shell model have the same scaling exponents of the corresponding linear advection models. If this statement were true, then it would allow to reduce the determination of the scaling exponents for the Navier-Stokes equation (1) to the easier problem of determination of the scaling exponents for the linear advection model (5). As far as the linear advection problem is concerned, its scaling exponents show anomalous behaviour (see among the others, [1, 2, 26] and the references therein).

It is then of interest to understand rigorously the properties of the joint system

$$(9) \quad \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u = -\nabla p_1 + \frac{\partial W_1}{\partial t}, \\ \frac{\partial w}{\partial t} - \nu \Delta w + (u \cdot \nabla)w = -\nabla p_2 + \frac{\partial W_2}{\partial t} \end{cases}$$

with the divergence free condition and the boundary condition. If it has a unique invariant measure μ , then the ensemble averages are done with respect to this measure, i.e. for the linear advection model (5)

$$(10) \quad S_w^p(l) = \iint \{[w(l\hat{\xi}) - w(0)] \cdot \hat{\xi}\}^p \mu(du, dw)$$

and for the Navier-Stokes equation (1)

$$(11) \quad S_u^p(l) = \iint \{[u(l\hat{\xi}) - u(0)] \cdot \hat{\xi}\}^p \mu(du, dw) \equiv \int \{[u(l\hat{\xi}) - u(0)] \cdot \hat{\xi}\}^p m(du)$$

being $m(du) = \int \mu(du, dw)$ the unique invariant measure for (1).

The analysis of system (9) has been performed by adding two terms:

$$(12) \quad \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \lambda(w \cdot \nabla)u = -\nabla p_1 + \frac{\partial W_1}{\partial t} \\ \frac{\partial w}{\partial t} - \nu \Delta w + (u \cdot \nabla)w + \lambda(w \cdot \nabla)w = -\nabla p_2 + \frac{\partial W_2}{\partial t} \end{cases}$$

where $\lambda \in \mathbb{R}$ is a parameter. We denote by (u^λ, w^λ) its solution. For $\lambda = 0$ we recover (9) and for $\lambda = 1$ system (12) is symmetric.

On the other hand, for any $\lambda \neq 0$, system (12) enjoys the following property: setting $v^\lambda = \lambda w^\lambda$ and multiplying the second equation by λ , we have a perfectly symmetric system for the pair (u^λ, v^λ) , except for the force and initial conditions:

$$(13) \quad \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + (v \cdot \nabla)u = -\nabla p_1 + \frac{\partial W_1}{\partial t} \\ \frac{\partial v}{\partial t} - \nu \Delta v + (u \cdot \nabla)v + (v \cdot \nabla)v = -\lambda \nabla p_2 + \lambda \frac{\partial W_2}{\partial t} \end{cases}$$

with $u^\lambda(0) = u_0$ and $v^\lambda(0) = \lambda w_0$. Thus, *assuming*¹ the universality of the scaling exponents to the force, it follows that u^λ and $v^\lambda = \lambda w^\lambda$ have the same scaling exponents for any $\lambda \neq 0$. Then the same consideration holds for the couple (u^λ, w^λ) for any $\lambda \neq 0$; indeed, $S_{\lambda w^\lambda}^p(l) = \lambda^p S_{w^\lambda}^p(l)$ and (8) gives the same ζ_p for v^λ and w^λ .

The crucial point is to see if this holds also in the limit as $\lambda \rightarrow 0$.

The continuous dependence of the solutions to system (12) as $\lambda \rightarrow 0$ has been investigated rigorously in the context of certain nonlinear phenomenological shell model (see [4] for the deterministic case and [5] for the stochastic case). Moreover, [5] consider the asymptotic dynamics (for large time) on the attractor,

¹The linear advection equation (5) has scaling exponents universal to the forcing, when the forces act only on large scales (this has been investigated by physicists, see e.g. [2, 9, 14]). Some numerical evidence shows the same for the nonlinear equation (1) (see [1]).

proving the continuous dependence on λ of the attractor. Here we address the problem for the stochastic 2D Navier-Stokes equation. We consider the stationary statistical regime described by an invariant measure. The main result of our paper is that the unique invariant measure μ^λ for system (12) converges to the unique invariant measure μ for system (9), as $\lambda \rightarrow 0$. Bearing in mind the expression (11), this implies that if the structure functions of u^λ and w^λ have the same scaling exponents for any $\lambda \neq 0$, this is true also for $\lambda = 0$, i.e. the Navier-Stokes equation (1) and the linear advection equation (5) have the same scaling exponents of the structure functions.

Finally, we point out that we present our results for the 2D Navier-Stokes equation, because of the lack of well-posedness in the 3D setting. But we do not use any peculiarity of the 2D case (except for the regularity results in the Appendix).

The paper is organized as follows. Section 2 gives the abstract setting and summarizes well known results on Navier-Stokes equation and Wiener processes. Classical results on the Markov semigroup are given in Section 4 and on the invariant measure μ^λ in Section 5. The new results are the continuous dependence of the solution on λ (Section 3) and the continuous dependence of the invariant measure on λ (Section 6). The Appendix provides regularity results, extending the results of the previous sections from the space of finite energy to more regular spaces involved in the definition of the structure functions.

2 Notations and hypothesis

2.1 Functional setting

We define the functional setting to study the Navier-Stokes equation. From now on, we consider a bidimensional spatial domain \mathcal{D} . We refer to [10, 31] for the main results.

Let \mathcal{V} be the space of infinitely differentiable vector fields u on \mathcal{D} with compact support strictly contained in \mathcal{D} , satisfying $\nabla \cdot u = 0$ in \mathcal{D} . We take the closure of this space in $[L^2(\mathcal{D})]^2$ and $[H_0^1(\mathcal{D})]^2$; we obtain respectively

$$H = \left\{ u \in [L^2(\mathcal{D})]^2 ; \nabla \cdot u = 0 \text{ in } \mathcal{D}, u \cdot n = 0 \text{ on } \partial\mathcal{D} \right\}$$

where n is the normal vector to $\partial\mathcal{D}$, and

$$V = \left\{ u \in [H^1(\mathcal{D})]^2 : \nabla \cdot u = 0 \text{ in } \mathcal{D}, u = 0 \text{ on } \partial\mathcal{D} \right\}.$$

They are separable Hilbert spaces with the inner products and norms inherited from $[L^2(\mathcal{D})]^2$ and $[H_0^1(\mathcal{D})]^2$ respectively:

$$|u|_H^2 = \langle u, u \rangle, \text{ and } \langle u, v \rangle = \int_{\mathcal{D}} [u(\xi) \cdot v(\xi)] d\xi,$$

$$\|u\|_V^2 = ((u, u)), \text{ and } ((u, v)) = \int_{\mathcal{D}} [\nabla u(\xi) \cdot \nabla v(\xi)] d\xi.$$

Denoting by H' and V' the dual spaces, if we identify H with H' , we have the Gelfand triple $V \subset H \subset V'$ with continuous dense injections. We denote

the dual pairing between $u \in V$ and $v \in V'$ by $\langle u, v \rangle_{V, V'}$. When $v \in H$, we have $\langle u, v \rangle_{V, V'} = \langle u, v \rangle$.

Let Π be the orthogonal projector in $[L^2(\mathcal{D})]^2$ onto H ; then the Stokes operator is

$$Au = -\Pi\Delta u, \quad \forall u \in D(A) = [H^2(\mathcal{D})]^2 \cap V.$$

The operator A is a closed positive unbounded self-adjoint operator in H with the inverse A^{-1} which is a self-adjoint compact operator in H ; by the classical spectral theorems there exists a sequence $\{\gamma_j\}_{j=1}^\infty$ of eigenvalues of the Stokes operator with $0 < \gamma_1 \leq \gamma_2 \leq \dots$, corresponding to the eigenvectors $e_j \in D(A)$; $\{e_j\}_j$ form an orthonormal basis in H . We have that γ_j behaves like j for $j \rightarrow \infty$.

For $\alpha > 0$ we will denote the α -power of the operator A by A^α and its domain by $D(A^\alpha)$; we have $\|u\|_{D(A^\alpha)}^2 = \sum_{j=1}^\infty \gamma_j^{2\alpha} |\langle u, e_j \rangle|^2$. Moreover, $V = D(A^{1/2})$, and $D(A^{\alpha_1})$ is compactly embedded in $D(A^{\alpha_2})$ for $\alpha_1 > \alpha_2$. By interpolation we have $\|u\|_{D(A^{1/4})} \leq C|u|_H^{1/2}\|u\|_V^{1/2}$. Finally, $D(A^{-\alpha})$ denotes the dual of $D(A^\alpha)$.

Let $b(\cdot, \cdot, \cdot) : V \times V \times V \rightarrow \mathbb{R}$ be the continuous trilinear form defined as

$$b(u, v, z) = \int_{\mathcal{D}} ([u(\xi) \cdot \nabla]v(\xi)) \cdot z(\xi) d\xi.$$

It is well known that there exists a continuous bilinear operator $B(\cdot, \cdot) : V \times V \rightarrow V'$ such that $\langle B(u, v), z \rangle = b(u, v, z)$, for all $z \in V$. By the incompressibility condition, for $u, v, z \in V$ we have (see e.g. [31])

$$(14) \quad \langle B(u, v), z \rangle_{V, V'} = -\langle B(u, z), v \rangle_{V, V'} \quad \text{and} \quad \langle B(u, v), v \rangle_{V, V'} = 0.$$

Furthermore, there exists a constant C such that

$$(15) \quad \|B(u, v)\|_{V'} \leq C\|u\|_{D(A^{1/4})}\|v\|_{D(A^{1/4})}.$$

This comes from Hölder inequality so $\|B(u, v)\|_{V'} \leq \|u\|_{L^4}\|v\|_{L^4}$, and then using the continuous embeddings $D(A^{1/4}) \subset [H^{1/2}(\mathcal{D})]^2 \subset [L^4(\mathcal{D})]^2$.

Projecting the equations of (12) onto H , the following abstract formulation is obtained

$$(16) \quad \begin{cases} du^\lambda + [\nu Au^\lambda + B(u^\lambda, u^\lambda) + \lambda B(w^\lambda, u^\lambda)]dt = dW_1 \\ dw^\lambda + [\nu Aw^\lambda + B(u^\lambda, w^\lambda) + \lambda B(w^\lambda, w^\lambda)]dt = dW_2 \\ u^\lambda(t_0) = u_0 \\ w^\lambda(t_0) = w_0 \end{cases}$$

with initial conditions $u_0, w_0 \in H$ at $t_0 \in \mathbb{R}$.

The Cauchy problem is studied on $[t_0, \infty[$.

Define $\tilde{H} = H \times H$, $\tilde{V} = V \times V$ and $D(\tilde{A}^\alpha) = D(A^\alpha) \times D(A^\alpha)$. If $x = (x_1, x_2) \in \tilde{H}$ and $y = (y_1, y_2) \in \tilde{H}$, we define the scalar product in \tilde{H} as

$$\langle x, y \rangle_{\tilde{H}} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$$

and the norms in \tilde{H} and \tilde{V} as

$$|x|_{\tilde{H}}^2 = |x_1|_H^2 + |x_2|_H^2, \quad x = (x_1, x_2) \in \tilde{H}$$

$$\|x\|_{\tilde{V}}^2 = \|x_1\|_{\tilde{V}}^2 + \|x_2\|_{\tilde{V}}^2 \quad x = (x_1, x_2) \in \tilde{V}.$$

Moreover, define the linear operator $\tilde{A} : D(\tilde{A}) \subset \tilde{H} \rightarrow \tilde{H}$, or also $\tilde{A} : \tilde{V} \rightarrow \tilde{V}'$, as $\tilde{A}x = (Ax_1, Ax_2)$; we have $\tilde{V} = D(\tilde{A}^{1/2})$.

For every $\lambda \in \mathbb{R}$, define the bilinear continuous operator \tilde{B}_λ from $\tilde{V} \times \tilde{V}$ to \tilde{V}' as

$$\tilde{B}_\lambda(x, y) = \left(B(x_1, y_1) + \lambda B(x_2, y_1), B(x_1, y_2) + \lambda B(x_2, y_2) \right),$$

where as usual we have used the notation $x = (x_1, x_2)$, $y = (y_1, y_2)$.

By (14)-(15), we have

Lemma 2.1 *For any λ*

$$\langle \tilde{B}_\lambda(x, y), z \rangle_{\tilde{V}, \tilde{V}'} = -\langle \tilde{B}_\lambda(x, z), y \rangle_{\tilde{V}, \tilde{V}'}, \quad \langle \tilde{B}_\lambda(x, y), y \rangle_{\tilde{V}, \tilde{V}'} = 0.$$

Moreover, there is a constant $C_\lambda > 0$ such that

$$\|\tilde{B}_\lambda(x, y)\|_{\tilde{V}'} \leq C_\lambda \|x\|_{D(\tilde{A}^{1/4})} \|y\|_{D(\tilde{A}^{1/4})}, \quad \forall x, y \in D(\tilde{A}^{1/4}).$$

We write (16) in more compact form as

$$(17) \quad \begin{cases} d\tilde{u}^\lambda + [\nu \tilde{A} \tilde{u}^\lambda + \tilde{B}_\lambda(\tilde{u}^\lambda, \tilde{u}^\lambda)] dt = d\tilde{W}, \\ \tilde{u}^\lambda(t_0) = \tilde{u}_0 \end{cases}$$

where $\tilde{u}^\lambda = (u^\lambda, w^\lambda)$, $\tilde{u}_0 = (u_0, w_0)$, $\tilde{W} = (W_1, W_2)$.

2.2 Stochastic driving force

The r.h.s. of the stochastic equations in (16) and (17) are defined as follows:

$$(18) \quad W_i(t) = \sum_{j=1}^{\infty} \sqrt{q_j} \beta_j^i(t) e_j,$$

with (β_j^i) mutually independent standard (scalar) two-sided Wiener processes defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and $q_j \geq 0$ for all j . This means that W_1 and W_2 are two independent two-sided Wiener processes with covariance operator $Q = \text{diag}\{q_j\}$; Q is a positive linear operator in the Hilbert space H .

Let us assume that the operator Q satisfies the following property: for some $\alpha_0 \in \mathbb{R}$

$$(19) \quad \sum_{j=1}^{\infty} q_j \gamma_j^{2\alpha_0} < \infty.$$

Then W_i is a $D(A^{\alpha_0})$ -valued Wiener process. For details we refer to [11]; but let us show the easy estimate

$$\mathbb{E}[\|W_i(t)\|_{D(A^{\alpha_0})}^2] = \mathbb{E}[\langle \sum_j \sqrt{q_j} \beta_j^i(t) A^{\alpha_0} e_j, \sum_h \sqrt{q_h} \beta_h^i(t) A^{\alpha_0} e_h \rangle] = \sum_j q_j \gamma_j^{2\alpha_0} |t|.$$

In the following we are interested in two cases:

- only a finite number of q_j do not vanish, i.e.

$$\exists n \in \mathbb{N} \text{ such that } q_j = 0 \quad \forall j > n.$$

In this case condition (19) is fulfilled for any α_0 .

- an infinite number of q_j do not vanish. In this case, the sum in (19) is a series and the sequence $\{q_j\}$ must satisfy $q_j \lesssim j^{-2\alpha_0-1-\varepsilon}$ for some $\varepsilon > 0$ as $j \rightarrow \infty$; by this we mean

$$\lim_{j \rightarrow \infty} q_j j^{2\alpha_0+1+\varepsilon} < \infty$$

(recall that $\gamma_j \lesssim j$ and $j \lesssim \gamma_j$ as $j \rightarrow \infty$). Notice that if condition (19) is fulfilled for a value α_0 then it holds for any value smaller than α_0 .

In the sequel, we consider $\alpha_0 \geq -\frac{1}{4}$ in (19). If $\alpha_0 = -\frac{1}{4}$ then the processes W_1 and W_2 have paths in $C(\mathbb{R}; D(A^{-1/4}))$ P-a.s. If $\alpha_0 = 0$ then Q is a trace class operator, i.e. $Tr Q := \sum_{j=1}^{\infty} q_j < \infty$.

The case of finite dimensional noise is the interesting one in turbulence theory (see [24]) and in numerical simulations: the force is injected at large scales (j small) and is transported by the non linearity to small scales (j large). However, the case of non degenerate (or full) noise, i.e. $q_j \neq 0$ for all j , is interesting from the mathematical point of view, in the sense that for some hydrodynamical models the unique invariant measure for system (13) is equivalent to the Gaussian invariant measure of the associated linear Ornstein-Uhlenbeck system. This has been proved in [18] for hyperviscous fluids, and can be proved for shell models following the setting of [3, 19]. This may be useful to get concrete expression for the invariant measure for system (13) (dealing with hyperviscous fluids or shell models). By the way this could help in checking the universality of the scaling exponents to the force. However, the results of [18] give so far the expression of the Radon-Nykodim derivative, obtained by Girsanov transform, only for the solution process on any finite time interval and not for the stationary process (even if a formal expression is given in [25] for Navier-Stokes and applies also to our model (13)).

3 Continuous dependence of the solution on λ

We begin giving the usual definition of solution of (17). This is a strong solution in the probabilistic sense and a weak (or generalized) solution in the sense of PDE's.

Definition 3.1 *We are given a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, a $D(\tilde{A}^{-1/4})$ -valued Wiener process \tilde{W} , a time interval $[t_0, T]$ and an initial data $\tilde{u}_0 \in \tilde{H}$. We say that a stochastic process \tilde{u}^λ is a solution of the Cauchy problem (17) on $[t_0, T]$ if it is a continuous \mathcal{F}_t -adapted process in \tilde{H} ,*

$$\tilde{u}^\lambda \in C([t_0, T]; \tilde{H}) \cap L^2(t_0, T; D(\tilde{A}^{1/4})) \quad P - a.s.$$

and (17) is satisfied in the following sense:

$$\begin{aligned} (20) \quad & \langle \tilde{u}^\lambda(t), \psi \rangle + \int_{t_0}^t \nu \langle \tilde{A}^{1/4} \tilde{u}^\lambda(s), \tilde{A}^{3/4} \psi \rangle ds - \int_{t_0}^t \langle \tilde{B}_\lambda(\tilde{u}^\lambda(s), \psi), \tilde{u}^\lambda(s) \rangle ds \\ & = \langle \tilde{u}_0, \psi \rangle + \langle \tilde{A}^{-1/4} \tilde{W}(t), \tilde{A}^{1/4} \psi \rangle - \langle \tilde{A}^{-1/4} \tilde{W}(t_0), \tilde{A}^{1/4} \psi \rangle \quad P - a.s. \end{aligned}$$

for any $t \in [t_0, T]$ and $\psi \in D(\tilde{A}^{3/4})$.

Recalling Lemma 2.1 we notice that

$$\langle \tilde{B}_\lambda(\tilde{u}^\lambda, \tilde{u}^\lambda), \psi \rangle = -\langle \tilde{B}_\lambda(\tilde{u}^\lambda, \psi), \tilde{u}^\lambda \rangle$$

and the latter expression is well defined. Therefore equation (20) corresponds to (17).

The well posedness of the stochastic Navier-Stokes equation is a well established result; we cannot give a complete list of references (see e.g. references in [20]). But we refer to the results proved for the 2D case with additive noise. Indeed, following [21, 17], one can prove existence and uniqueness of the solution to system (17). The procedure is the same as for the Navier-Stokes equation, since the operator \tilde{B}_λ enjoys the same properties of the bilinear operator B of Navier-Stokes, as given in Lemma 2.1.

Theorem 3.2 *Let $\lambda \in \mathbb{R}$. Let us assume that $\tilde{u}_0 \in \tilde{H}$ and that assumption (19) is satisfied with $\alpha_0 = -\frac{1}{4}$. Then, there is a process \tilde{u}^λ with paths in $C([t_0, T]; \tilde{H}) \cap L^4(t_0, T; D(\tilde{A}^{1/4}))$ P -a.s., which is the unique solution \tilde{u}^λ for (17) in the sense of Definition 3.1.*

In the proof, one introduces the Ornstein-Uhlenbeck process \tilde{z} solution of the (linear) Stokes type problem $d\tilde{z} + \nu\tilde{A}\tilde{z} dt = d\tilde{W}$, $\tilde{z}(t_0) = 0$:

$$\tilde{z}(t) = \int_{t_0}^t e^{-\nu\tilde{A}(t-s)} d\tilde{W}(s).$$

We have (see [11])

Proposition 3.3 *Let assumption (19) be satisfied. Then, the process \tilde{z} is a well defined Gaussian, ergodic and continuous process in $D(\tilde{A}^{\alpha_0 + \frac{1}{2}})$. In particular, for any finite interval $[t_0, T]$ there is a P -a.s. finite random variable $\tilde{C} = \tilde{C}(\omega)$ such that*

$$(21) \quad \sup_{t_0 \leq t \leq T} \|\tilde{z}(t, \omega)\|_{D(\tilde{A}^{\alpha_0 + \frac{1}{2}})} \leq \tilde{C}(\omega) \quad \text{for } P - \text{a.e. } \omega.$$

In particular, for $\alpha_0 = -\frac{1}{4}$ we have $\tilde{z} \in C(\mathbb{R}; D(\tilde{A}^{1/4}))$ P -a.s.

The aim of this section is to prove the continuous dependence of the solution \tilde{u}^λ on λ .

Theorem 3.4 *Let assumption (19) be fulfilled with $\alpha_0 = -\frac{1}{4}$. Then, for any $\lambda_0 \in \mathbb{R}$, $\tilde{u}_0 \in \tilde{H}$ and finite time interval $[t_0, T]$, we have*

$$\lim_{\lambda \rightarrow \lambda_0} \sup_{t_0 \leq t \leq T} |\tilde{u}^\lambda(t) - \tilde{u}^{\lambda_0}(t)|_{\tilde{H}} = 0 \quad P - \text{a.s.}$$

Proof. We prove the theorem only in the case $\lambda_0 = 0$ and $t_0 = 0$, the general case being the same. Since $\alpha_0 = -\frac{1}{4}$ we have that $\tilde{z} \in C([0, T]; D(\tilde{A}^{1/4}))$ P -a.s., and there exists a random variable \tilde{C} (which is finite P -a.s.) such that

$$(22) \quad \|\tilde{z}\|_{C([0, T]; D(\tilde{A}^{1/4}))} \leq \tilde{C} \quad P - \text{a.s.}$$

Step 1 (preparation). Let us define the new process

$$q^\lambda := u^\lambda + \lambda w^\lambda$$

and the corresponding difference

$$\rho^\lambda := q^\lambda - q^0 = q^\lambda - u^0.$$

Since u^λ and w^λ satisfy

$$du^\lambda(t) + [\nu Au^\lambda(t) + B(u^\lambda(t), u^\lambda(t)) + \lambda B(w^\lambda(t), u^\lambda(t))] dt = dW_1(t)$$

$$dw^\lambda(t) + [\nu Aw^\lambda(t) + B(u^\lambda(t), w^\lambda(t)) + \lambda B(w^\lambda(t), w^\lambda(t))] dt = dW_2(t)$$

then q^λ and ρ^λ are solutions, respectively, of

$$dq^\lambda(t) + [\nu Aq^\lambda(t) + B(q^\lambda(t), q^\lambda(t))] dt = dW_1(t) + \lambda dW_2(t)$$

$$d\rho^\lambda(t) + [\nu A\rho^\lambda(t) + B(q^\lambda(t), \rho^\lambda(t)) + B(\rho^\lambda(t), q^\lambda(t)) + B(\rho^\lambda(t), \rho^\lambda(t))] dt = \lambda dW_2(t).$$

These equations are obtained by using the bilinearity of the operator B and have to be understood in the integral sense (in time) and with scalar product with suitable test functions, as in Definition 3.1.

Step 2 (bound on q^λ). Let us prove next that P -a.s.

$$(23) \quad \sup_{\lambda \in [-1, 1]} \sup_{0 \leq t \leq T} |q^\lambda(t)|_H < \infty,$$

and

$$(24) \quad \sup_{\lambda \in [-1, 1]} \int_0^T \|q^\lambda(t)\|_{D(A^{1/4})}^4 dt < \infty.$$

Notice that the equation satisfied by q^λ is the Navier-Stokes equation with initial data $u_0 + \lambda w_0$ and forcing term $dW_1 + \lambda dW_2$. Therefore we get a priori estimates following the technique of [21]. We define

$$\bar{v}^\lambda := q^\lambda - z^\lambda,$$

where z^λ is the solution of the linear problem

$$dz^\lambda + \nu Az^\lambda dt = dW_1 + \lambda dW_2, \quad z^\lambda(0) = 0.$$

In this way we get rid of the stochastic force. Indeed \bar{v}^λ satisfies P -a.s. a deterministic equation (no stochastic integral appears):

$$\frac{d}{dt} \bar{v}^\lambda(t) + \nu A \bar{v}^\lambda(t) + B(\bar{v}^\lambda(t) + z^\lambda(t), \bar{v}^\lambda(t) + z^\lambda(t)) = 0.$$

As we shall see, the paths of the process \bar{v}^λ are more regular than those of q^λ and z^λ . We are going to prove that $\bar{v}^\lambda \in C([0, T]; H) \cap L^2(0, T; V)$ P -a.s.. From now on in the proof we proceed pathwise.

Formally, taking the scalar product in H with \bar{v}^λ and using (14) we get

$$\begin{aligned} \frac{1}{2} |\bar{v}^\lambda(t)|_H^2 + \nu \int_0^t \|\bar{v}^\lambda(s)\|_V^2 ds &\leq \frac{1}{2} |u_0 + \lambda w_0|_H^2 \\ &+ \int_0^t |\langle B(\bar{v}^\lambda(s) + z^\lambda(s), \bar{v}^\lambda(s)), z^\lambda(s) \rangle| ds. \end{aligned}$$

Rigorously, the above inequality can be proved either by general abstract theorems or by taking finite-dimensional approximations, performing the computations at the finite dimensional level and then taking the limit (see [31]). Then we have

$$\begin{aligned}
(25) \quad & \frac{1}{2} |\bar{v}^\lambda(t)|_H^2 + \nu \int_0^t \|\bar{v}^\lambda(s)\|_V^2 ds - \frac{1}{2} |u_0 + \lambda w_0|_H^2 \\
& \leq \int_0^t |\langle B(\bar{v}^\lambda(s) + z^\lambda(s)), \bar{v}^\lambda(s) \rangle| ds \\
& \leq C \int_0^t |\bar{v}^\lambda(s)|_H^{1/2} \|\bar{v}^\lambda(s)\|_V^{3/2} \|z^\lambda(s)\|_{D(A^{1/4})} ds + C \int_0^t \|\bar{v}^\lambda(s)\|_V \|z^\lambda(s)\|_{D(A^{1/4})}^2 ds \\
& \leq \frac{\nu}{2} \int_0^t \|\bar{v}^\lambda(s)\|_V^2 ds + \frac{C_\nu}{2} \int_0^t |\bar{v}^\lambda(s)|_H^2 \|z^\lambda(s)\|_{D(A^{1/4})}^4 ds + \frac{C_\nu}{2} \int_0^t \|z^\lambda(s)\|_{D(A^{1/4})}^4 ds.
\end{aligned}$$

From now on, C_ν denotes a constant depending on ν , which may vary from line to line. Hence,

$$|\bar{v}^\lambda(t)|_H^2 \leq |u_0 + \lambda w_0|_H^2 + C_\nu \|z^\lambda\|_{C([0,T];D(A^{1/4}))}^4 \int_0^t |\bar{v}^\lambda(s)|_H^2 ds + C_\nu t \|z^\lambda\|_{C([0,T];D(A^{1/4}))}^4.$$

By Gronwall lemma we deduce

$$(26) \quad |\bar{v}^\lambda(t)|_H^2 \leq \left(|u_0 + \lambda w_0|_H^2 + 1 \right) e^{C_\nu \|z^\lambda\|_{C([0,T];D(A^{1/4}))}^4 t} \quad \forall t \in [0, T];$$

thus

$$\sup_{\lambda \in [-1,1]} \sup_{0 \leq t \leq T} |\bar{v}^\lambda(t)|_H < \infty \quad P - a.s.$$

Since $q^\lambda = \bar{v}^\lambda + z^\lambda$, this implies (23), using the bound (22) on the paths of z^λ .

From (25) and (26) we get

$$\sup_{\lambda \in [-1,1]} \int_0^T \|\bar{v}^\lambda(t)\|_V^2 dt < \infty,$$

and therefore

$$\sup_{\lambda \in [-1,1]} \int_0^T \|\bar{v}^\lambda(t)\|_{D(A^{1/4})}^4 dt \leq \left(\sup_{\lambda \in [-1,1]} \sup_{0 \leq t \leq T} |\bar{v}^\lambda(t)|_H^2 \right) \left(\sup_{\lambda \in [-1,1]} \int_0^T \|\bar{v}^\lambda(t)\|_V^2 dt \right) < \infty.$$

From this (24) follows.

Step 3 (convergence of ρ^λ). Next we prove that

$$(27) \quad \lim_{\lambda \rightarrow 0} \sup_{0 \leq t \leq T} |\rho^\lambda(t)|_H = 0,$$

and

$$(28) \quad \lim_{\lambda \rightarrow 0} \int_0^T \|\rho^\lambda(t)\|_{D(A^{1/4})}^4 dt = 0.$$

In one sentence, this is a consequence of the various bounds that we have established previously, and the fact that the initial condition λw_0 and the forcing term $\lambda W_2(t)$ converge to zero, as $\lambda \rightarrow 0$.

As before, define the difference

$$\widehat{v}^\lambda := \rho^\lambda - \lambda z,$$

where

$$z(t) = \int_0^t e^{-\nu A(t-s)} dW_2(s)$$

is the solution of $dz(t) + \nu Az(t)dt = dW_2(t)$, $z(0) = 0$. We work on \widehat{v}^λ in a similar way we did for \widehat{v}^λ . Then

$$\frac{d}{dt} \widehat{v}^\lambda(t) + \nu A \widehat{v}^\lambda(t) + B(q^\lambda(t), \rho^\lambda(t)) + B(\rho^\lambda(t), q^\lambda(t)) - B(\rho^\lambda(t), \rho^\lambda(t)) = 0.$$

By (14) and (15), taking the scalar product with \widehat{v}^λ in H we have

$$\begin{aligned} |\langle B(q^\lambda, \rho^\lambda), \widehat{v}^\lambda \rangle| &= |\langle B(q^\lambda, \widehat{v}^\lambda), \rho^\lambda \rangle| = |\langle B(q^\lambda, \widehat{v}^\lambda), \lambda z \rangle| \\ &\leq C \|q^\lambda\|_{D(A^{1/4})} \|\widehat{v}^\lambda\|_V \|\lambda z\|_{D(A^{1/4})} \\ &\leq \frac{\nu}{6} \|\widehat{v}^\lambda\|_V^2 + C_\nu \lambda^2 \|q^\lambda\|_{D(A^{1/4})}^4 + C_\nu \lambda^2 \|z\|_{D(A^{1/4})}^4. \end{aligned}$$

In a similar way we estimate

$$\begin{aligned} |\langle B(\rho^\lambda, q^\lambda), \widehat{v}^\lambda \rangle| &= |\langle B(\rho^\lambda, \widehat{v}^\lambda), q^\lambda \rangle| = |\langle B(\widehat{v}^\lambda + \lambda z, \widehat{v}^\lambda), q^\lambda \rangle| \\ &\leq |\langle B(\widehat{v}^\lambda, \widehat{v}^\lambda), q^\lambda \rangle| + |\langle B(\lambda z, \widehat{v}^\lambda), q^\lambda \rangle| \\ &\leq \frac{\nu}{6} \|\widehat{v}^\lambda\|_V^2 + C_\nu \|q^\lambda\|_{D(A^{1/4})}^4 |\widehat{v}^\lambda|_H^2 + C_\nu \lambda^2 \|q^\lambda\|_{D(A^{1/4})}^4 + C_\nu \lambda^2 \|z\|_{D(A^{1/4})}^4 \end{aligned}$$

and

$$\begin{aligned} |\langle B(\rho^\lambda, \rho^\lambda), \widehat{v}^\lambda \rangle| &= |\langle B(\rho^\lambda, \widehat{v}^\lambda), \rho^\lambda \rangle| = |\langle B(\widehat{v}^\lambda + \lambda z, \widehat{v}^\lambda), \lambda z \rangle| \\ &\leq |\langle B(\widehat{v}^\lambda, \widehat{v}^\lambda), \lambda z \rangle| + \lambda^2 |\langle B(z, \widehat{v}^\lambda), z \rangle| \\ &\leq \frac{\nu}{6} \|\widehat{v}^\lambda\|_V^2 + C_\nu \lambda^4 \|z\|_{D(A^{1/4})}^4 |\widehat{v}^\lambda|_H^2 + C_\nu \lambda^4 \|z\|_{D(A^{1/4})}^4. \end{aligned}$$

Hence

$$\begin{aligned} |\widehat{v}^\lambda(t)|_H^2 + \nu \int_0^t \|\widehat{v}^\lambda(s)\|_V^2 ds &\leq \lambda^2 |w_0|_H^2 + C_\nu \int_0^t \left(\|q^\lambda\|_{D(A^{1/4})}^4 + \lambda^4 \|z\|_{D(A^{1/4})}^4 \right) |\widehat{v}^\lambda(s)|_H^2 ds \\ &\quad + C_\nu \lambda^4 \|z\|_{D(A^{1/4})}^4 + C_\nu \lambda^2 \left(\|q^\lambda\|_{D(A^{1/4})}^4 + \|z\|_{D(A^{1/4})}^4 \right). \end{aligned}$$

As usual, by Gronwall lemma we get

$$\lim_{\lambda \rightarrow 0} \sup_{0 \leq t \leq T} |\widehat{v}^\lambda(t)|_H = 0;$$

in addition

$$\lim_{\lambda \rightarrow 0} \int_0^T \|\widehat{v}^\lambda(s)\|_V^2 ds = 0.$$

As in the previous step, they imply the claims (27) and (28).

Step 4 (convergence of w^λ). We define $\xi^\lambda := w^\lambda - w^0$, and prove that

$$(29) \quad \lim_{\lambda \rightarrow 0} \sup_{0 \leq t \leq T} |\xi^\lambda(t)|_H = 0, \quad P - a.s.;$$

this is the statement of the theorem for the second component of \tilde{u}^λ .

We have

$$\begin{aligned} \frac{d}{dt}\xi^\lambda(t) + \nu A\xi^\lambda(t) &= B(u^0(t), w^0(t)) - B(u^\lambda(t), w^\lambda(t)) - \lambda B(w^\lambda(t), w^\lambda(t)) \\ &= B(q^0(t), w^0(t)) - B(q^\lambda(t), w^\lambda(t)) \\ &= -B(\rho^\lambda(t), w^0(t)) - B(q^\lambda(t), \xi^\lambda(t)). \end{aligned}$$

As in the previous steps, we deduce that

$$\begin{aligned} |\xi^\lambda(t)|_H^2 + \nu \int_0^t \|\xi^\lambda(s)\|_V^2 ds &\leq \int_0^t |\langle B(\rho^\lambda(s), \xi^\lambda(s)), w^0(s) \rangle| ds \\ &\leq \frac{\nu}{2} \int_0^t \|\xi^\lambda(s)\|_V^2 ds + C_\nu \int_0^t \|\rho^\lambda(s)\|_{D(A^{1/4})}^2 \|w^0(s)\|_{D(A^{1/4})}^2 ds. \end{aligned}$$

Hence,

$$|\xi^\lambda(t)|_H^2 \leq C_\nu \left(\int_0^t \|w^0(s)\|_{D(A^{1/4})}^4 ds \right)^{1/2} \left(\int_0^t \|\rho^\lambda(s)\|_{D(A^{1/4})}^4 ds \right)^{1/2}.$$

Using (28) we get (29).

Step 5 (convergence of u^λ). We simply notice that

$$|u^\lambda(t) - u^0(t)|_H = |q^\lambda(t) - q^0(t) - \lambda w^\lambda(t)|_H = |\rho^\lambda(t) - \lambda w^\lambda(t)|_H;$$

therefore, by the results of steps 3 and 4, we have

$$\lim_{\lambda \rightarrow 0} \sup_{0 \leq t \leq T} |u^\lambda(t) - u^0(t)|_H = 0.$$

The proof is complete. \square

4 Transition semigroup

Let us denote by $\tilde{u}^\lambda(t; x)$ the solution of (17) with initial condition x , by $\mathcal{B}_b(\tilde{H})$ the space of Borel bounded functions $\varphi : \tilde{H} \rightarrow \mathbb{R}$ and by $\mathcal{C}_b(\tilde{H})$ its subspace of continuous bounded functions. Define the operators $P_t^\lambda : \mathcal{B}_b(\tilde{H}) \rightarrow \mathcal{B}_b(\tilde{H})$ as

$$(30) \quad (P_t^\lambda \varphi)(x) = \mathbb{E}[\varphi(\tilde{u}^\lambda(t; x))].$$

The process \tilde{u}^λ is said to be a Markov process in \tilde{H} if

$$(31) \quad \mathbb{E}[\varphi(\tilde{u}^\lambda(t+s; x)) | \mathcal{F}_t] = (P_s^\lambda \varphi)(\tilde{u}^\lambda(t; x)) \quad P - \text{a.s.}$$

for any $\varphi \in \mathcal{C}_b(\tilde{H})$, $t, s > 0$, $x \in \tilde{H}$. Taking expectation in (31) one obtains the semigroup property $P_{t+s}^\lambda = P_t^\lambda P_s^\lambda$ on $\mathcal{C}_b(\tilde{H})$.

The Markov semigroup $\{P_t^\lambda\}_t$ is said to be Feller in \tilde{H} if

$$P_t^\lambda : \mathcal{C}_b(\tilde{H}) \rightarrow \mathcal{C}_b(\tilde{H}) \quad \forall t.$$

There is an extensive literature for the 2D Navier-Stokes equation driven by an additive noise. In particular we refer to [20], Section 3.4, for the proof of the Markov property and to [21] for the proof of the Feller property. Notice that the proofs of [20, 21] for the 2D stochastic Navier-Stokes equation apply also to our system (17). Therefore we have

Proposition 4.1 *With the same assumption of Theorem 3.2, the solution \tilde{u}^λ of system (17) defines a Markov semigroup which is Feller in \tilde{H} .*

As a consequence of the continuous dependence of the solution \tilde{u}^λ on λ given in Theorem 3.4, we have the following result.

Corollary 4.2 *With the same assumptions of Theorem 3.4, for any $t, \lambda_0 \in \mathbb{R}$ we have*

$$(32) \quad \lim_{\lambda \rightarrow \lambda_0} P_t^\lambda \varphi(x) = P_t^{\lambda_0} \varphi(x), \quad \forall \varphi \in \mathcal{C}_b(\tilde{H}), \quad x \in \tilde{H}.$$

Proof. Keeping in mind definition (30), this is a consequence of Theorem 3.4 and the Lebesgue dominated convergence theorem. \square

5 Invariant measure

A probability measure μ^λ is invariant for P_t^λ in \tilde{H} if

$$(33) \quad \int_{\tilde{H}} P_t^\lambda \varphi(x) d\mu^\lambda(x) = \int_{\tilde{H}} \varphi(x) d\mu^\lambda(x), \quad \forall \varphi \in \mathcal{B}_b(\tilde{H}), \quad \forall t.$$

We restrict to $\varphi \in \mathcal{C}_b(\tilde{H})$, since the Markov semigroup is Feller.

For any $\lambda \in \mathbb{R}$ one can prove existence and uniqueness of the invariant measure for system (17), following the lines of the proofs for the 2D Navier-Stokes equation. We briefly recall the results.

It is not difficult to prove the existence of an invariant measure for the semigroup P_t^λ . Indeed, existence is proved assuming again (19) with $\alpha_0 = -\frac{1}{4}$ (see [21] Theorem 3.1). For the sake of simplicity we sketch the proof of the existence following [8], which requires the operator \tilde{Q} to be trace class, i.e. $\alpha_0 = 0$.

First, use Itô formula on the process $|\tilde{u}^\lambda(t)|_{\tilde{H}}^2$

$$(34) \quad \begin{aligned} d|\tilde{u}^\lambda(t)|_{\tilde{H}}^2 &= -2\nu \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^2 dt - 2\langle \tilde{B}(\tilde{u}^\lambda(t), \tilde{u}^\lambda(t)), \tilde{u}^\lambda(t) \rangle dt \\ &\quad + 2\langle \tilde{u}^\lambda(t), d\tilde{W} \rangle + Tr\tilde{Q} dt, \end{aligned}$$

where $Tr\tilde{Q} = 2TrQ$. Then use (14), integrate in time and take expectation

$$(35) \quad \mathbb{E}|\tilde{u}^\lambda(t)|_{\tilde{H}}^2 + 2\nu \int_0^t \mathbb{E}\|\tilde{u}^\lambda(s)\|_{\tilde{V}}^2 ds = |x|_{\tilde{H}}^2 + 2(TrQ)t.$$

Define the family of measures $\{m_T^\lambda\}_{T>0}$ on Borel subsets Γ of \tilde{H}

$$m_T^\lambda(\Gamma) := \frac{1}{T} \int_0^T P\{\tilde{u}^\lambda(t; 0) \in \Gamma\} dt.$$

Since (35) implies

$$2\nu \int_0^T \mathbb{E} \|\tilde{u}^\lambda(t; 0)\|_{\tilde{V}}^2 dt \leq 2(Tr Q) T ,$$

then

$$(36) \quad \int \|x\|_{\tilde{V}}^2 dm_T^\lambda(x) \leq \frac{Tr Q}{\nu} \quad \text{for any } T > 0.$$

By Chebyshev inequality

$$m_T^\lambda(x : \|x\|_{\tilde{V}}^2 > R) \leq \frac{1}{R} \frac{Tr Q}{\nu},$$

that is the family of measures $\{m_T^\lambda\}_{T>0}$ is tight in any space which contains \tilde{V} as compact subset. Let us say that it is tight in \tilde{H} . Now, thanks to Feller property we can apply Krylov-Bogoliubov method to get that there exists at least one invariant measure μ^λ for system (17) (see e.g. [12, 20]). This μ^λ is the weak limit of a subsequence of m_T^λ ; moreover (36) implies

$$(37) \quad \int \|x\|_{\tilde{V}}^2 d\mu^\lambda(x) \leq \frac{Tr Q}{\nu}.$$

Remark 5.1 *Inequality (37) is indeed an equality, i.e.*

$$(38) \quad \int \|x\|_{\tilde{V}}^2 d\mu^\lambda(x) = \frac{Tr Q}{\nu}$$

for any invariant measure μ^λ obtained as the weak limit of a subsequence of $\{m_T^\lambda\}$ in the Krylov-Bogoliubov method. Indeed, let u_{st}^λ be the stationary process solving (17) and whose law at any fixed time is μ^λ . Then (35) becomes

$$\mathbb{E} |\tilde{u}_{st}^\lambda(t)|_{\tilde{H}}^2 + 2\nu \int_0^t \mathbb{E} \|\tilde{u}_{st}^\lambda(s)\|_{\tilde{V}}^2 ds = \mathbb{E} |\tilde{u}_{st}^\lambda(0)|_{\tilde{H}}^2 + 2(Tr Q)t$$

and (37) implies that each term is well defined. By stationarity, we get

$$2\nu \int_0^t \mathbb{E} \|\tilde{u}_{st}^\lambda(s)\|_{\tilde{V}}^2 ds = 2(Tr Q)t, \quad \forall t > 0;$$

hence (38).

The question of uniqueness of invariant measures is more subtle. In the last twenty years many results on uniqueness appeared; see, among the others, [6, 7, 15, 13, 23, 27, 29, 30].

The best result of uniqueness of the invariant measure has been proved by means of the irreducibility and asymptotic strong Feller property by Hairer and Mattingly [27] assuming that the noise acts on first few modes, i.e.

$$(HQf) \quad \exists m \in \mathbb{N}, \exists j_1, j_2, \dots, j_m \in \mathbb{N} \text{ such that} \\ q_{j_1}, q_{j_2}, \dots, q_{j_m} \neq 0 \text{ and the remaining } q_j = 0.$$

Actually the number m and the indices j_1, j_2, \dots, j_m have to be chosen in a proper way, but the choice is independent of the viscosity. In particular, one

can take $m = 4$ when the spatial domain \mathcal{D} is a box and periodic boundary conditions are assumed (see [27]); general domains can be considered in the same way (see [28]).

On the other hand, the first result of uniqueness of invariant measure for the stochastic 2D Navier-Stokes equation has been proved by Flandoli and Maslowski [23] assuming irreducibility and strong Feller property (see [12] for this technique for general SPDE's). This requires the noise to be full ($q_j \neq 0$ for all j) and with suitable space regularity

$$\exists \varepsilon > 0 : D(A^{1/2}) \subseteq R(\sqrt{Q}) \subseteq D(A^{\frac{3}{8}+\varepsilon})$$

$R(\sqrt{Q})$ being the range of the operator \sqrt{Q} . Since $D(A^{1/2}) \subseteq R(\sqrt{Q})$ is incompatible with $\text{Tr} Q < \infty$, we write the assumption on the covariance operator Q in the more general form given in [15], which allows Q to be a trace class operator:

$$(HQi) \quad \exists \varepsilon > 0 \text{ and } p \in]\frac{1}{4}, \frac{1}{2}[: D(A^{2p}) \subseteq R(\sqrt{Q}) \subseteq D(A^{\frac{1}{4}+\frac{p}{2}+\varepsilon}).$$

For instance, one can take $q_j = Cj^{-a}$ with $1 < a < 2$.

From now on, we assume (HQf) or (HQi) in order to get the uniqueness of the invariant measure μ^λ for system (17).

Combining the existence and uniqueness results we get

Theorem 5.2 *Assume $\text{Tr} Q < \infty$.*

If (HQf) or (HQi) hold, then for any $\lambda \in \mathbb{R}$ system (17) has a unique invariant measure μ^λ . Moreover it is ergodic, i.e.

$$(39) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T P_t^\lambda \varphi(x) dt = \int_{\tilde{H}} \varphi d\mu^\lambda, \quad \forall \varphi \in \mathcal{C}_b(\tilde{H}) \quad \text{and} \quad \forall x \in \tilde{H}.$$

and

$$\int \|x\|_{\tilde{V}}^2 d\mu^\lambda(x) = \frac{\text{Tr} Q}{\nu}.$$

Finally, we have that the family $\{\mu^\lambda\}_{\lambda \in \mathbb{R}}$ of these (unique) invariant measures is tight in \tilde{H} , that is

$$(40) \quad \forall \epsilon > 0 \exists K_\epsilon \text{ compact subset of } \tilde{H} : \inf_{\lambda \in \mathbb{R}} \mu^\lambda(K_\epsilon) > 1 - \epsilon.$$

Theorem 5.3 *With the assumptions of Theorem 5.2 we have that the family $\{\mu^\lambda\}_{\lambda \in \mathbb{R}}$ is tight in \tilde{H} .*

Proof. Let us consider (38) and denote by $K_R = \left\{x \in \tilde{V} : \|x\|_{\tilde{V}}^2 \leq R\right\}$ and by K_R^c its complementary; K_R is a compact ball in \tilde{H} . Then

$$\mu^\lambda(K_R^c) \leq \int \frac{\|x\|_{\tilde{V}}^2}{R} \mu^\lambda(dx) \leq \frac{\text{Tr} Q}{\nu} \frac{1}{R}.$$

Tightness follows choosing $R = \frac{\text{Tr} Q}{\epsilon \nu}$ in (40). \square

6 Continuous dependence of the invariant measure on λ

In this section we prove our main result on the invariant measures μ^λ .

Theorem 6.1 *With the assumptions of Theorem 5.2 we have that μ^λ is weakly convergent to μ^{λ_0} , whenever λ converges to λ_0 .*

Proof. We prove the theorem only in the case $\lambda_0 = 0$, the general case being the same. We previously proved that the family $\{\mu^\lambda\}_\lambda$ is tight in \tilde{H} . By Prokhorov's theorem, any sequence $\{\mu^{\lambda^{(k)}}\}_{k \in \mathbb{N}}$ with $\lim_{k \rightarrow \infty} \lambda^{(k)} = 0$ has a subsequence $\{\mu^{\lambda^{(k_j)}}\}_{j \in \mathbb{N}}$ weakly convergent to some measure β :

$$(41) \quad \lim_{j \rightarrow \infty} \int \varphi d\mu^{\lambda^{(k_j)}} = \int \varphi d\beta \quad \forall \varphi \in \mathcal{C}_b(\tilde{H}).$$

We need to prove that $\beta = \mu^0$ independently of the chosen sequence.

By using the fact that the measure μ^λ and μ^0 are the unique invariant measures with respect to P_t^λ and P_t^0 respectively and (39) holds, then for any $\varphi \in \mathcal{C}_b(\tilde{H})$ we have for any $x \in \tilde{H}$

$$\begin{aligned} \int_{\tilde{H}} \varphi d\mu^\lambda &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T P_t^\lambda \varphi(x) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (P_t^\lambda \varphi(x) - P_t^0 \varphi(x)) dt + \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T P_t^0 \varphi(x) dt \\ &= \mathcal{E}_x^\lambda + \int_{\tilde{H}} \varphi d\mu^0. \end{aligned}$$

Now, we use Corollary 4.2 and that for any $\lambda \in \mathbb{R}$ and $x \in \tilde{H}$ we have $|P_t^\lambda \varphi(x)| \leq \|\varphi\|_{\mathcal{C}_b}$. By means of the dominated convergence theorem, we infer that for any $x \in \tilde{H}$

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \mathcal{E}_x^\lambda &= \lim_{\lambda \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (P_t^\lambda \varphi(x) - P_t^0 \varphi(x)) dt \\ &= \lim_{T \rightarrow \infty} \lim_{\lambda \rightarrow 0} \frac{1}{T} \int_0^T (P_t^\lambda \varphi(x) - P_t^0 \varphi(x)) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \lim_{\lambda \rightarrow 0} (P_t^\lambda \varphi(x) - P_t^0 \varphi(x)) dt \\ &= 0. \end{aligned}$$

Now, recalling (41) we get that

$$(42) \quad \int_{\tilde{H}} \varphi d\beta = \int_{\tilde{H}} \varphi d\mu^0$$

for any arbitrary $\varphi \in \mathcal{C}_b(\tilde{H})$. Thus, $\beta = \mu^0$. \square

A Regularity results

We have proved our results working in the space \tilde{H} of finite energy. But one needs the velocity vector field u to be regular enough in space and p -integrable in order to define the structure functions (10)-(11). Here we summarize regularity results, providing that each measure μ^λ has support contained in $[C(\mathcal{D})]^2$ and every p -moment is finite.

In the previous sections the results on the continuous dependence of the solution and of the invariant measure on λ have been proved in \tilde{H} , but can be extended in the same way in more regular spaces.

Following Appendix 1 of [22] we have

Theorem A.1 *Let \tilde{u}_0 be a \mathcal{F}_0 -measurable random variable and $p \geq 2$.*

i) If $\mathbb{E}[\|\tilde{u}_0\|_{\tilde{H}}^p] < \infty$ and $TrQ < \infty$, then there exists a unique solution \tilde{u}^λ for system (17) such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\tilde{u}^\lambda(t)|_{\tilde{H}}^p \right] + \nu p \mathbb{E} \int_0^T |\tilde{u}^\lambda(t)|_{\tilde{H}}^{p-2} \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^2 dt \leq C(p, T, \mathbb{E}|\tilde{u}^\lambda(0)|_{\tilde{H}}^p, TrQ).$$

ii) Let the spatial domain \mathcal{D} be a torus.

If $\mathbb{E}[\|\tilde{u}_0\|_{\tilde{V}}^p] < \infty$ and $Tr(AQ) < \infty$, then there exists a unique solution \tilde{u}^λ for system (17) such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^p \right] + \nu p \mathbb{E} \int_0^T \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^{p-2} \|\tilde{u}^\lambda(t)\|_{D(\tilde{A})}^2 dt \leq C(p, T, \mathbb{E}\|\tilde{u}^\lambda(0)\|_{\tilde{V}}^p, Tr(AQ)).$$

The basic tool to prove this result is Itô formula on the process $|\tilde{u}^\lambda(t)|_{\tilde{H}}^p$

$$\begin{aligned} d|\tilde{u}^\lambda(t)|_{\tilde{H}}^p &= -\nu p |\tilde{u}^\lambda(t)|_{\tilde{H}}^{p-2} \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^2 dt - p |\tilde{u}^\lambda(t)|_{\tilde{H}}^{p-2} \langle \tilde{B}_\lambda(\tilde{u}^\lambda(t), \tilde{u}^\lambda(t)), \tilde{u}^\lambda(t) \rangle dt \\ &\quad + p |\tilde{u}^\lambda(t)|_{\tilde{H}}^{p-2} \langle \tilde{u}^\lambda(t), d\tilde{W} \rangle + \frac{1}{2} p(p-1) |\tilde{u}^\lambda(t)|_{\tilde{H}}^{p-2} 2TrQ dt. \end{aligned}$$

or on the process $\|\tilde{u}^\lambda(t)\|_{\tilde{V}}^p$

$$\begin{aligned} d\|\tilde{u}^\lambda(t)\|_{\tilde{V}}^p &= -\nu p \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^{p-2} \|\tilde{u}^\lambda(t)\|_{D(\tilde{A})}^2 dt - p \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^{p-2} \langle \tilde{B}_\lambda(\tilde{u}^\lambda(t), \tilde{u}^\lambda(t)), \tilde{A}\tilde{u}^\lambda(t) \rangle dt \\ &\quad + p \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^{p-2} \langle \tilde{A}\tilde{u}^\lambda(t), d\tilde{W} \rangle + \frac{1}{2} p(p-1) \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^{p-2} 2Tr(AQ) dt. \end{aligned}$$

The latter relationship is studied by using $\langle \tilde{B}_\lambda(x, x), \tilde{A}x \rangle = 0$ which holds only when the spatial domain \mathcal{D} is a box and periodic boundary conditions are assumed; indeed, this property comes from the conservation of vorticity for an inviscid fluid.

A consequence of this theorem is that for any deterministic initial data in \tilde{H} , when $TrQ < \infty$ for the solution \tilde{u}^λ one has

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\tilde{u}^\lambda(t)|_{\tilde{H}}^p \right] + \nu p \mathbb{E} \int_0^T |\tilde{u}^\lambda(t)|_{\tilde{H}}^{p-2} \|\tilde{u}^\lambda(t)\|_{\tilde{V}}^2 dt < \infty.$$

Therefore for any invariant measure

$$\int |x|_{\tilde{H}}^p d\mu^\lambda(dx) < \infty, \quad \int |x|_{\tilde{H}}^{p-2} \|x\|_{\tilde{V}}^{p-2} d\mu^\lambda(dx) < \infty.$$

In a similar way we work under the assumption $Tr(AQ) < \infty$.

Finally, we generalize the result (38) for the invariant measures.

Theorem A.2 *Let $p \geq 2$.*

i) If $TrQ < \infty$, then

$$\int |x|_{\tilde{H}}^{p-2} \|x\|_{\tilde{V}}^2 \mu^\lambda(dx) = \frac{p-1}{\nu} (TrQ) \int |x|_{\tilde{H}}^{p-2} \mu^\lambda(dx).$$

ii) If \mathcal{D} is a torus and $Tr(AQ) < \infty$, then

$$\int \|x\|_{\tilde{V}}^{p-2} \|x\|_{D(\tilde{A})}^2 \mu^\lambda(dx) = \frac{p-1}{\nu} (Tr(AQ)) \int \|x\|_{\tilde{V}}^{p-2} \mu^\lambda(dx).$$

To prove this result it is enough to work on the stationary process whose law at fixed time is μ^λ and use the previous identities from Itô formula.

We point out that when the operator Q is full, the uniqueness of the invariant measure can be proved under the assumption $Tr(AQ) < \infty$ using the results of [16] (a further generalization of (HQi)).

Item *ii*) with $p = 2$ of the latter theorem implies that

$$\int \|x\|_{D(\tilde{A})}^2 \mu^\lambda(dx) < \infty;$$

so the measure μ^λ has support in $D(\tilde{A})$.

We conclude working on the torus, to get the p -integrability of the $[C(\mathcal{D})]^2$ -norm. When \mathcal{D} is a torus and $Tr(AQ) < \infty$, let C_γ be the constant such that $\|x\|_{\tilde{V}} \leq C_\gamma \|x\|_{D(\tilde{A})}$. Then we have

$$\int \|x\|_{\tilde{V}}^{p-2} \|x\|_{D(\tilde{A})}^2 \mu^\lambda(dx) \leq \frac{(p-1)!!}{\nu^{\frac{p}{2}}} Tr(Q) [Tr(AQ)]^{\frac{p}{2}-1} C_\gamma^{p-4}$$

for any p even and greater than 2. This is proved by induction using the relationships of Theorem A.2. Now, by interpolation, for any $p > 2$ we have $\|x\|_{D(\tilde{A}^{\frac{1}{2} + \frac{1}{p}})}^p \leq C_p \|x\|_{D(\tilde{A}^{\frac{1}{2}})}^{p-2} \|x\|_{D(\tilde{A})}^2$; then using that $H^{1+\epsilon}(\mathcal{D}) \subset C(\mathcal{D})$ for any $\epsilon > 0$, we get

$$\int \|x\|_{[C(\mathcal{D})]^2}^p \mu^\lambda(dx) \leq \overline{C}_p \int \|x\|_{\tilde{V}}^{p-2} \|x\|_{D(\tilde{A})}^2 \mu^\lambda(dx) < \infty.$$

Therefore the integrals appearing in the structure functions

$$S_{u^\lambda}^p(l) = \int \{[x_1(l\hat{\xi}) - x_1(0)] \cdot \hat{\xi}\}^p \mu^\lambda(dx),$$

$$S_{w^\lambda}^p(l) = \int \{[x_2(l\hat{\xi}) - x_2(0)] \cdot \hat{\xi}\}^p \mu^\lambda(dx)$$

are well defined for any $p \geq 2$ when the spatial domain \mathcal{D} is a torus and $Tr(AQ) < \infty$.

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